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The boundary-element method for the determination of a heat source dependent on one variable

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Abstract. This paper investigates the inverse problem of determining a heat source in the parabolic heat equation using the usual conditions of the direct problem and a supplementary condition, called an overdetermination. In this problem, if the heat source is taken to be space-dependent only, then the overdetermination is the temperature measurement at a given single instant, whilst if the heat source is time-dependent only, then the overdetermination is the transient temperature measurement recorded by a single thermocouple installed in the interior of the heat conductor. These measurements ensure that the inverse problem has a unique solution, but this solution is unstable, hence the problem is ill-posed. This instability is overcome using the Tikhonov regularization method with the discrepancy principle or the L-curve criterion for the choice of the regularization parameter. The boundary-element method (BEM) is developed for solving numerically the inverse problem and numerical results for some benchmark test examples are obtained and discussed.

Key words: boundary-element method, heat source, inverse problems, parabolic equation

1. Introduction

The inverse problem of determining an unknown heat-source function in the heat-conduction equation has been considered in many theoretical papers, notably [1–4]. With the exception of [4], where the source is sought as a function of both space and time but is additive or separable, in all the other studies the source has been sought as a function of space or time only. However, no numerical implementations have been attempted yet under such a rigorous mathematical back-up. In this paper the determinination of the unknown heat source is sought from the usual conditions of the direct problem and a supplementary condition, called an overdetermination. Two inverse problems are formulated in Section 2. In the first problem we take the heat source to be time-dependent only and the overdetermination is the transient temperature measured at a single interior point of the space domain, whilst in the second problem the heat source is space-dependent only and the overdetermination is the temperature measurements along the domain at a given single instant. Although sufficient conditions for the unique solvability of the inverse problem are provided, the problem is still ill-posed since small errors, inherently present in any practical measurement, give rise to unbounded and highly oscillatory solutions. One approach to solve this problem, which is referred to in the literature as the method of output least squares, is to assume that the unknown heat-source function is of a specific functional form depending on some parameters and then seek to determine optimal parameter values which minimize an error functional based on the over-specified data. However, this approach has the drawback that it is usually not evident that the solution of the optimization problem solves the original inverse problem. Therefore, in this paper, in order to overcome the instability of the solution, the BEM combined with the

Tikhonov regularization and the discrepancy principle or the L-curve criterion for the choice of the regularization parameter is developed (see Section 3) for the numerical solution of the two inverse problems formulated in Section 2. Section 4 discusses the numerical results for two benchmark test examples involving a spacewise and a timewise heat source. While solutions of the linear heat equation with various kinds of source terms are widely known in the literature, see *e.g.* [5–9], an original contribution of the paper may be found in the numerical treatment of two inverse problems in order to obtain the stability of the numerical solution under random perturbations of the input data. Finally, conclusions, some extensions and future work are presented in Section 5.

2. Formulation of the inverse problems

Let $T>0, \alpha \in (0,1)$ and $l>0$ be fixed numbers and let us consider first the one-dimensional time-dependent problem in which the source $Q(x,t)=f(t)$ depends on time only. Hereafter we use the Sobolev functional space notation of [10]; see the Appendix.

2.1. Inverse problem I

Find the temperature $u \in H^{2+\alpha,1+\alpha/2}([0,l]\times[0,T])$ and the heat source $f \in H^{\alpha/2}([0,T])$ which satisfy the heat-conduction equation with a time-dependent source, namely

$$\frac{\partial u}{\partial t}(x,t)=\frac{\partial^2 u}{\partial x^2}(x,t)+f(t), \qquad (x,t)\in(0,l)\times(0,T], \tag{1}$$

subject to the boundary conditions

$$u(0,t)=h_0(t), \quad u(l,t)=h_l(t), \qquad t\in[0,T], \tag{2}$$

the initial condition

$$u(x,0)=u_0(x), \qquad x\in[0,l] \tag{3}$$

and the overspecified condition

$$u(x_0,t)=\chi(t), \qquad t\in[0,T], \tag{4}$$

where $x_0\in(0,l)$ is the interior location of a thermocouple recording the temperature measurement (4) and $h_0(t), h_l(t), u_0(x)$ and $\chi(t)$ are given functions.

We assume that the conditions (2–4) are consistent up to the first order, *i.e.*,

$$u_0(0)=h_0(0), \quad u_0(l)=h_l(0), \quad u_0(x_0)=\chi(0), \tag{5}$$

$$h_0'(0)=u_0''(0)+\chi'(0)-u_0''(x_0), \quad h_l'(0)=u_0''(l)+\chi'(0)-u_0''(x_0). \tag{6}$$

Then the unique solvability of the inverse problem (1)–(4) follows from the following theorem; see Prilepko and Solov'ev [2].

Theorem 1. *If* $h_0, h_l, \chi \in H^{1+\alpha/2}([0,T]), u_0\in H^{2+\alpha}([0,l])$*, and the conditions* (2)–(4) *are consistent up to the first order, as given in* (5) *and* (6)*, then the problem* (1)–(4) *has a unique solution* $(u,f)\in H^{2+\alpha,1+\alpha/2}([0,l]\times[0,T])\times H^{\alpha/2}([0,T])$.

If instead of the condition (4) we specify a heat flux data, namely

$$-\frac{\partial u}{\partial x}(0,t)=q_0(t), \qquad t\in[0,T], \tag{7}$$

where $q_0(t)$ is a given function, we have the following solvability theorem; see [11].

Theorem 2. *If* $h_0 = h_l \equiv 0$, $u_0 \in H_0^1(0, l)$ *and* $q_0 \in C^1(0, T)$ *then the inverse problem* (1)–(3) *and* (7) *has a unique solution* $(u, f) \in (H_0^1((0, l) \times (0, T)) \cap H^2((0, l) \times (0, T))) \times C(0, T)$.

Theorems 1 and 2 show that the inverse problems (1)–(4) and (1)–(3), (7), respectively, have unique solutions, but they are ill-posed since their solutions do not depend continuously on the input data.

We now consider the problem in which the source $Q(x, t) = g(x)$ depends on space only.

2.2. Inverse problem II

Find the temperature $u \in H^{2+\alpha, 1+\alpha/2}([0, l] \times [0, T])$ and the heat source $g \in H^{\alpha}([0, l])$ which satisfy the heat-conduction equation with a space-dependent heat source, namely

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + g(x), \qquad (x, t) \in (0, l) \times (0, T], \tag{8}$$

subject to the boundary and initial conditions (2) and (3), respectively, and the overspecified condition

$$u(x, T) = \Psi(x), \qquad x \in [0, l]. \tag{9}$$

We assume that the conditions (2), (3) and (9) are consistent up to the first order, *i.e.*,

$$u_0(0) = h_0(0), \quad u_0(l) = h_l(0), \quad h_0(T) = \Psi(0), \quad h_l(T) = \Psi(l), \tag{10}$$

$$h_0'(0) - u_0''(0) = h_0''(T) - \Psi''(0), \quad h_l'(0) - u_0''(l) = h_l''(T) - \Psi''(l). \tag{11}$$

Then the unique solvability of the inverse problem (2), (3), (8) and (9) follows from the following theorem.

Theorem 3. *If* $h_0, h_l \in H^{1+\alpha/2}([0, T])$, $u_0, \Psi \in H^{2+\alpha}([0, l])$, *and the conditions* (2), (3) *and (9) are consistent up to the first order, then the problem* (2), (3), (8) *and* (9) *has a unique solution* $(u, g) \in Q := H^{2+\alpha, 1+\alpha/2}([0, l] \times [0, T]) \times H^{\alpha}([0, l])$.

Proof. From [3] we know that the operator of the inverse problem (2), (3), (8) and (9) is Fredholm with zero index and therefore it suffices to prove that the problem given by (8) with the homogeneous conditions (2), (3) and (9), *i.e.*,

$$u(0, t) = u(l, t) = 0, \quad t \in [0, T], \tag{12}$$

$$u(x, 0) = u(x, T) = 0, \quad x \in [0, l] \tag{13}$$

has only the trivial solution. By introducing the function $v(x, t) = u(x, t) + F(x)$, where

$$F(x) = \int_0^x \int_0^{x'} g(x'')\mathrm{d}x''\mathrm{d}x' - \frac{x}{l}\int_0^l \int_0^{x'} g(x'')\mathrm{d}x''\mathrm{d}x', \tag{14}$$

we have that v satisfies the problem

$$\frac{\partial v}{\partial t}(x, t) = \frac{\partial^2 v}{\partial x^2}(x, t), \tag{15}$$

$$v(0, t) = v(l, t) = 0, \quad t \in [0, T], \tag{16}$$

$$v(x, 0) - v(x, T) = 0, \quad x \in [0, l]. \tag{17}$$

Using Fourier sine-series, we may write the solution of the problem (15)–(17) as

$$v(x,t)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \mathrm{e}^{-\frac{n^2\pi^2 t}{l^2}}, \tag{18}$$

where

$$b_n=\frac{2}{\pi}\int_0^{\pi} v(x,0)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x \tag{19}$$

and

$$v(x,T)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \mathrm{e}^{-\frac{n^2\pi^2 T}{l^2}}=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)=v(x,0), \qquad x\in[0,l]. \tag{20}$$

From (20) it follows that $b_n=0$ for $n\geq 1$, *i.e.*, $v\equiv 0$. Then $u(x,t)+F(x)=0$, which from (13) implies that $u\equiv 0$ and $F\equiv 0$. By differentiating twice (14) we have that $g\equiv 0$. Hence, the unique solvability of the inverse problem (2), (3), (8) and (9) is established.

At this stage it is worth noting that instead of the "upper base" condition (9), one can specify the heat-flux data, namely

$$-\frac{\partial u}{\partial x}(0,t)=q_0(t), \quad \text{or} \quad \frac{\partial u}{\partial x}(l,t)=q_l(t) \qquad t\in[0,T]. \tag{21}$$

However, in this case the existence of a solution for the inverse problem (2), (3), (8) and (21) is not guaranteed, although the uniqueness holds; see [5].

If the boundary conditions (2) are homogeneous, *i.e.*, $h_0=h_l\equiv 0$, then we also have the unique solvability of the inverse problem (2), (3), (8) and (9); see [11].

Theorem 4. *If* $h_0=h_l\equiv 0$ *and* $u_0, \Psi\in H_0^1(0,l)\cap H^2(0,l)$ *then the inverse problem* (2), (3), (8) *and* (9) *has a unique solution* $(u,g)\in(H_0^1((0,l)\times(0,T))\cap H^2((0,l)\times(0,T)))\times L^2(0,l)$.

Although the Inverse Problem II has a unique solution, it is still ill-posed since the solution does not depend continuously on the input data.

At this stage, we can remark that the two problems I and II have a unique stable component of the solution, $u(x,t)$, and a unique, but unstable, component of the solution, $f(t)$ or $g(x)$. Therefore, in order to overcome this instability of the solution in the heat-source component we employ the BEM combined with the Tikhonov regularization technique, as described in the next section.

3. The boundary-element method

By applying Green's formula we can recast Equation (1) or (8) in the integral form

$$\begin{aligned}\eta(x)u(x,t)=&\int_0^t \left[G(x,t,\xi,\tau)\frac{\partial u}{\partial n(\xi)}(\xi,\tau)-u(\xi,\tau)\frac{\partial G}{\partial n(\xi)}(x,t,\xi,\tau)\right]\Bigg|_{\xi=0}^{\xi=l}\mathrm{d}\tau\\ &+\int_0^l G(x,t,y,0)u(y,0)\mathrm{d}y+\int_0^l\int_0^t G(x,t,y,\tau)Q(y,\tau)\mathrm{d}\tau\mathrm{d}y\\ &\text{for } (x,t)\in[0,l]\times(0,T],\end{aligned} \tag{22}$$

where $\eta(0)=\eta(l)=1/2, \eta(x)=1$ for $x\in(0,l), n$ is the outward normal to the space boundary $\{0,l\}\times[0,T]$, *i.e.*, $n(0)=-1$ and $n(l)=1, Q(y,\tau)=f(\tau)$ for Equation (1) and $Q(y,\tau)=g(y)$

for Equation (8), and G is the fundamental solution of the one-dimensional heat equation, namely

$$G(x,t,y,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi(t-\tau)}}\exp\left(-\frac{(x-y)^2}{4(t-\tau)}\right), \tag{23}$$

where H is the Heaviside step function.

Let us now discretize each of the boundaries $\{0\}\times[0,T]$ and $\{l\}\times[0,T]$ into N equidistant boundary elements $[t_{i-1},t_i]$ for $i=\overline{1,N}$, $t_i=iT/N$ for $i=\overline{0,N}$, and the space interval $[0,l]$ into N_0 equidistant cells, $[x_{k-1},x_k]$ for $k=\overline{1,N_0}$, $x_k=kl/N_0$ for $k=\overline{0,N_0}$.

Using a constant BEM, we assume that the function u and its normal derivative $\frac{\partial u}{\partial n}$ are constant over each interval and we take the value at their midpoints $\tilde{t_i}=(t_i+t_{i-1})/2=(2i-1)T/N$ for $i=\overline{1,N}$, and $\tilde{x_k}=(x_k+x_{k-1})/2=(2k-1)l/N_0$ for $k=\overline{1,N_0}$, namely

$$u(0,t)=h_0(\tilde{t_i}):=h_{0i},\quad u(l,t)=h_l(\tilde{t_i}):=h_{li}, \tag{24}$$

$$\frac{\partial u}{\partial n}u(0,t)=\frac{\partial u}{\partial n}(0,\tilde{t_i}):=q_{0i},\quad \frac{\partial u}{\partial n}u(l,t)=\frac{\partial u}{\partial n}(l,\tilde{t_i}):=q_{li}, \tag{25}$$

$$u(x,0)=u_0(\tilde{x_k}):=u_{0k}, \tag{26}$$

for $t\in[t_{i-1},t_i)$, $i=\overline{1,N}$ and $x\in[x_{k-1},x_k)$, $k=\overline{1,N_0}$.

With these approximations, the integral equation (22) can be approximated as

$$\begin{aligned}\eta(x)u(x,t)=&\sum_{j=1}^{N}\left[A_{0j}(x,t)q_{0j}+A_{lj}(x,t)q_{lj}-B_{0j}(x,t)h_{0j}-B_{lj}(x,t)h_{lj}\right]+\\&+\sum_{k=1}^{N_0}C_k(x,t)u_{0,k}+D(x,t),\end{aligned} \tag{27}$$

where

$$\begin{aligned}&A_{0j}(x,t)=\int_{t_{j-1}}^{t_j}G(x,t,0,\tau)\mathrm{d}\tau,\quad A_{lj}(x,t)=\int_{t_{j-1}}^{t_j}G(x,t,l,\tau)\mathrm{d}\tau,\\&B_{0j}(x,t)=\int_{t_{j-1}}^{t_j}\frac{\partial G}{\partial n(0)}(x,t,0,\tau)\mathrm{d}\tau,\quad B_{lj}(x,t)=\int_{t_{j-1}}^{t_j}\frac{\partial G}{\partial n(l)}(x,t,l,\tau)\mathrm{d}\tau,\\&C_k(x,t)=\int_{x_{k-1}}^{x_k}G(x,t,y,0)\mathrm{d}y,\quad \text{for } j=\overline{1,N},\quad k=\overline{1,N_0},\end{aligned} \tag{28}$$

$$D(x,t)=\int_0^l\int_0^t G(x,t,y,\tau)Q(y,\tau)\mathrm{d}\tau\mathrm{d}y. \tag{29}$$

Special attention is paid now to the domain integral (29). We seek a piecewise constant approximation for the heat source $Q(y,\tau)$ and therefore we assume that

$$f(t)=f(\tilde{t_i}):=f_i,\quad t\in[t_{i-1},t_i),\quad i=\overline{1,N}, \tag{30}$$

$$g(x)=g(\tilde{x_k}):=g_k,\quad x\in[x_{k-1},x_k),\quad i=\overline{1,N_0}. \tag{31}$$

The expression (29) is therefore approximated as

$$D(x,t)=\int_0^t f(\tau)\left(\int_0^l G(x,t,y,\tau)\mathrm{d}y\right)\mathrm{d}\tau=\sum_{j=1}^{N}D_j^I(x,t)f_j, \tag{32}$$

if $Q(x,t)=f(t)$ and

$$D(x,t)=\int_0^l g(y)\left(\int_0^t G(x,t,y,\tau)\mathrm{d}\tau\right)\mathrm{d}y=\sum_{k=1}^{N_0} D_k^{II}(x,t)g_k, \tag{33}$$

if $Q(x,t)=g(x)$, where

$$D_j^I(x,t)=\int_{t_{j-1}}^{t_j}\int_0^l G(x,t,y,\tau)\mathrm{d}y\mathrm{d}\tau, \quad j=\overline{1,N}, \tag{34}$$

$$D_k^{II}(x,t)=\int_{x_{k-1}}^{x_k}\int_0^t G(x,t,y,\tau)\mathrm{d}\tau\mathrm{d}y, \quad k=\overline{1,N_0}. \tag{35}$$

The integrals in Equations (28), (34) and (35) can be evaluated analytically and their expressions are given in the Appendix.

By applying the integral equation (27) at the boundary nodes $(0,\tilde{t}_i)$ and $(l,\tilde{t}_i)$ for $i=\overline{1,N}$, we obtain the system of $2N$ equations

$$\mathbf{Aq}-\mathbf{Bh}+\mathbf{Cu}_0=\mathbf{d} \tag{36}$$

where

$$A_{i,j}=A_{0j}(0,\tilde{t}_i), \quad A_{i,(j+N)}=A_{lj}(0,\tilde{t}_i), \quad i,j=\overline{1,N}, \tag{37}$$

$$A_{(i+N),j}=A_{0j}(l,\tilde{t}_i), \quad A_{(i+N),(j+N)}=A_{lj}(l,\tilde{t}_i), \quad i,j=\overline{1,N}, \tag{38}$$

$$B_{i,j}=B_{0j}(0,\tilde{t}_i)+0.5\delta_{ij}, \quad B_{i,(j+N)}=B_{lj}(0,\tilde{t}_i), \quad i,j=\overline{1,N}, \tag{39}$$

$$B_{(i+N),j}=B_{0j}(l,\tilde{t}_i), \quad B_{(i+N),(j+N)}=B_{lj}(l,\tilde{t}_i)+0.5\delta_{ij}, \quad i,j=\overline{1,N}, \tag{40}$$

$$q_j=q_{0j}, \quad q_{j+N}=q_{lj}, \quad h_j=h_{0j}, \quad h_{j+N}=h_{lj}, \quad j=\overline{1,N}, \tag{41}$$

$$C_{i,k}=C_k(0,\tilde{t}_i), \quad C_{(i+N),k}=C_k(l,\tilde{t}_i), \quad k=\overline{1,N_0}, \quad i=\overline{1,N}, \tag{42}$$

$$d_i=-D(0,\tilde{t}_i), \quad d_{i+N}=-D(l,\tilde{t}_i), \quad k=\overline{1,N_0}, \quad i=\overline{1,N}, \tag{43}$$

where δ_{ij} is the Kronecker delta symbol.

Assuming that $\mathbf{A}$ is invertible, we can express the flux $\mathbf{q}$ from (36) as

$$\mathbf{q}=\mathbf{A}^{-1}\mathbf{Bh}-\mathbf{A}^{-1}\mathbf{Cu}_0-\mathbf{A}^{-1}\mathbf{d}. \tag{44}$$

We now use the conditions (4) or (9) by applying the integral equation (22) at the points $(x_0,\tilde{t}_i)$ for $i=\overline{1,N}$, or $(\tilde{x_m},T)$ for $m=\overline{1,N_0}$, to obtain the system of N equations

$$\boldsymbol{\chi}=\mathbf{A}^{(1)}\mathbf{q}-\mathbf{B}^{(1)}\mathbf{h}+\mathbf{C}^{(1)}\mathbf{u}_0+\mathbf{d}^{(1)} \tag{45}$$

or

$$\boldsymbol{\Psi}=\mathbf{A}^{(2)}\mathbf{q}-\mathbf{B}^{(2)}\mathbf{h}+\mathbf{C}^{(2)}\mathbf{u}_0+\mathbf{d}^{(2)}, \tag{46}$$

where

$$\chi_i=\chi(\tilde{t}_i), \quad \Psi_m=\Psi(\tilde{x_m}), \quad i=\overline{1,N} \quad m=\overline{1,N_0}, \tag{47}$$

$$A_{i,j}^{(1)}=A_{0j}(x_0,\tilde{t}_i), \quad A_{i,(j+N)}^{(1)}=A_{lj}(x_0,\tilde{t}_i), \quad i,j=\overline{1,N}, \tag{48}$$

$$B_{i,j}^{(1)}=B_{0j}(x_0,\tilde{t}_i), \quad B_{i,(j+N)}^{(1)}=B_{lj}(x_0,\tilde{t}_i), \quad i,j=\overline{1,N}, \tag{49}$$

$$C_{i,k}^{(1)}=C_k(x_0,\tilde{t}_i), \quad d_i^{(1)}=D(x_0,\tilde{t}_i), \quad i=\overline{1,N} \quad k=\overline{1,N_0}, \tag{50}$$

$$A_{m,j}^{(2)}=A_{0j}(\tilde{x_m},T), \quad A_{m,(j+N)}^{(2)}=A_{lj}(\tilde{x_m},T), \quad m=\overline{1,N_0} \quad j=\overline{1,N}, \tag{51}$$

$$B_{m,j}^{(2)}=B_{0j}(\tilde{x_m},T), \quad B_{m,(j+N)}^{(2)}=B_{lj}(\tilde{x_m},T), \quad m=\overline{1,N_0} \quad j=\overline{1,N}, \tag{52}$$

$$C_{m,k}^{(2)}=C_k(\tilde{x_m},T), \quad d_m^{(2)}=D(\tilde{x_m},T), \qquad m,k=\overline{1,N_0}. \tag{53}$$

For problem I the vectors $\mathbf{d}$ and $\mathbf{d}^{(1)}$ are given by

$$\mathbf{d} = \mathbf{D}^I \mathbf{f}, \qquad \mathbf{d}^{(1)} = \mathbf{D}^{(1)} \mathbf{f}, \tag{54}$$

where

$$D^I_{i,j} = D^I_j(0, \tilde{t}_i), \quad D^I_{(i+N),j} = D^I_j(l, \tilde{t}_i), \quad i, j = \overline{1, N}, \tag{55}$$

$$D^{(1)}_{i,j} = D^I_j(x_0, \tilde{t}_i), \quad i, j = \overline{1, N}, \tag{56}$$

whilst for Problem II the vectors $\mathbf{d}$ and $\mathbf{d}^{(2)}$ are given by

$$\mathbf{d} = \mathbf{D}^{II} \mathbf{g}, \qquad \mathbf{d}^{(2)} = \mathbf{D}^2 \mathbf{g}, \tag{57}$$

where

$$D^{II}_{i,k} = D^{II}_k(0, \tilde{t}_i), \quad D^{II}_{(i+N),k} = D^{II}_k(l, \tilde{t}_i), \quad i = \overline{1, N}, \quad k = \overline{1, N_0}, \tag{58}$$

$$D^2_{m,k} = D^{II}_k(\tilde{x_m}, T), \quad m, k = \overline{1, N_0}. \tag{59}$$

Based on the above notations we obtain

$$(\mathbf{D}^{(1)} - \mathbf{A}^{(1)}\mathbf{A}^{-1}\mathbf{D}^I)\mathbf{f} + (\mathbf{A}^{(1)}\mathbf{A}^{-1}\mathbf{B} - \mathbf{B}^{(1)})\mathbf{h} + (\mathbf{C}^{(1)} - \mathbf{A}^{(1)}\mathbf{A}^{-1}\mathbf{C})\mathbf{u}_0 = \boldsymbol{\chi} \tag{60}$$

for problem I, and

$$(\mathbf{D}^{(2)} - \mathbf{A}^{(2)}\mathbf{A}^{-1}\mathbf{D}^{II})\mathbf{g} + (\mathbf{A}^{(2)}\mathbf{A}^{-1}\mathbf{B} - \mathbf{B}^{(2)})\mathbf{h} + (\mathbf{C}^{(2)} - \mathbf{A}^{(2)}\mathbf{A}^{-1}\mathbf{C})\mathbf{u}_0 = \boldsymbol{\Psi} \tag{61}$$

for problem II.

By introducing the following notation:

$$\mathbf{z}_1 = \boldsymbol{\chi} + (\mathbf{B}^{(1)} - \mathbf{A}^{(1)}\mathbf{A}^{-1}\mathbf{B})\mathbf{h} + (\mathbf{A}^{(1)}\mathbf{A}^{-1}\mathbf{C} - \mathbf{C}^{(1)})\mathbf{u}_0, \tag{62}$$

$$\mathbf{z}_2 = \boldsymbol{\Psi} + (\mathbf{B}^{(2)} - \mathbf{A}^{(2)}\mathbf{A}^{-1}\mathbf{B})\mathbf{h} + (\mathbf{A}^{(2)}\mathbf{A}^{-1}\mathbf{C} - \mathbf{C}^{(2)})\mathbf{u}_0, \tag{63}$$

$$\mathbf{X}_1 = \mathbf{D}^{(1)} - \mathbf{A}^{(1)}\mathbf{A}^{-1}\mathbf{D}^I, \qquad \mathbf{X}_2 = \mathbf{D}^{(2)} - \mathbf{A}^{(2)}\mathbf{A}^{-1}\mathbf{D}^{II}, \tag{64}$$

we obtain that problem I has been reduced to solving the following $N \times N$ system of linear equations:

$$\mathbf{X}_1 f = \mathbf{z}_1, \tag{65}$$

whilst the problem II has been reduced to solving the following $N_0 \times N_0$ system of linear equations

$$\mathbf{X}_2 g = \mathbf{z}_2. \tag{66}$$

From Section 2 we know that the initial-boundary-value problems I and II have unique solutions. Although this is not a proof for the unique solvability of the discrete counterparts of the integral formulations of these problems, it is reasonable to assume here that these systems of linear equations have unique solutions. However, these systems are ill-conditioned since the inverse problems under consideration are ill-posed. Thus, if the right-hand sides of Equations (65) and (66) are contaminated with errors, *i.e.*,

$$||\mathbf{z}_1{}^{\epsilon} - \mathbf{z}_1|| \leq \epsilon, \quad ||\mathbf{z}_2{}^{\epsilon} - \mathbf{z}_2|| \leq \epsilon, \tag{67}$$

then the inverse solutions of (65) and (66), namely ${\bf X_1}^{-1}{\bf z_1}^{\epsilon}$ and ${\bf X_2}^{-1}{\bf z_2}^{\epsilon}$ will be very different from the exact solutions ${\bf X_1}^{-1}{\bf z}_1$ and ${\bf X_2}^{-1}{\bf z}_2$, respectively. Therefore, instead of the straightforward inversion of, say, (65) we use a stable regularization method given by (see [12]) the minimization of the Tikhonov functional

$$T_{\lambda}({\bf f}) = ({\bf X}_1 - {\bf z_1}^{\epsilon})^{\rm tr}({\bf X}_1 - {\bf z_1}^{\epsilon}) + \lambda({\bf Rf})^{\rm tr}({\bf Rf}), \tag{68}$$

where the superscript tr denotes the transpose of a matrix, λ is a regularization parameter, and the regularization matrices ${\bf R}^{tr}{\bf R}$ are given by (see [12,13])

$$ {\bf R}^{\rm tr}{\bf R} = \begin{bmatrix} 1\,0 & . \\ 0\,1 & . \\ . & . & . \end{bmatrix} \qquad \text{(zeroth-order regularization)}, \tag{69}$$

$$ {\bf R}^{\rm tr}{\bf R} = \begin{bmatrix} 1 & -1 & 0 & 0 & . \\ -1 & 2 & -1 & 0 & . \\ 0 & -1 & 2 & -1 & . \\ . & . & . & . & . \end{bmatrix} \qquad \text{(first-order regularization)}, \tag{70}$$

$$ {\bf R}^{\rm tr}{\bf R} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & . & . & . \\ -2 & 5 & -4 & 1 & 0 & . & . & . \\ 1 & -4 & 6 & -4 & 1 & 0 & . & . \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \qquad \text{(second-order regularization)}. \tag{71}$$

It should be observed that the first part of the Tikhonov functional (68) is a measure of the fit of the regularized solution to the measured data, while the second part of the functional is a measure of the smoothness of either the regularized solution, its first- or second-order derivative. Minimizing the Tikhonov functional (68), we obtain a solution **f** depending on λ, namely

$$ {\bf f}_{\lambda} = ({\bf X_1}^{\rm tr}{\bf X}_1 + \lambda{\bf R}^{\rm tr}{\bf R})^{-1}{\bf X_1}^{\rm tr}{\bf z_1}^{\epsilon}. \tag{72}$$

A similar solution can be obtained for the system (66), namely

$$ {\bf g}_{\mu} = ({\bf X_2}^{\rm tr}{\bf X}_2 + \mu{\bf R}^{\rm tr}{\bf R})^{-1}{\bf X_2}^{\rm tr}{\bf z_2}^{\epsilon}. \tag{73}$$

The regularization parameters λ and μ can be chosen according to the L-curve method; see [14]. Alternatively, if by plotting the norm of the residuals $||{\bf X}_1 f_{\lambda} - {\bf z_1}^{\epsilon}||$ and $||{\bf X}_2 g_{\mu} - {\bf z_2}^{\epsilon}||$ versus the solutions norms $||{\bf f}_{\lambda}||$ and $||{\bf g}_{\mu}||$, respectively, an L-curve is not obtained, then one may employ the discrepancy principle (see [15]) *i.e.*, we choose $\lambda, \mu > 0$ such that

$$ ||{\bf X}_1 f_{\lambda} - {\bf z_1}^{\epsilon}|| \approx \epsilon, \quad ||{\bf X}_2 g_{\mu} - {\bf z_2}^{\epsilon}|| \approx \epsilon. \tag{74}$$

4. Numerical results and discussion

In this section we present and discuss the numerical results obtained by employing the BEM combined with the Tikhonov regularization technique presented in Section 3, for two typical benchmark examples. For these examples we have taken $l = T = 1$ and $x_0 = 0{\cdot}5$. The number of boundary elements was taken $N = N_0 = 40$, which was found to be sufficiently large to ensure that any further increase in this discretization did not significantly affect the accuracy of the numerical solutions of the direct problems (1)–(3), or (2),(3),(8) if $f(t)$ and $g(x)$ were known. The choice of the regularization parameters λ or μ was based on the discrepancy principle or the L-curve criterion.

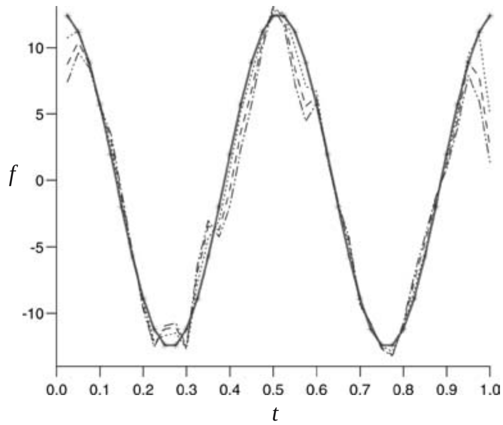

Figure 1. The numerical time-dependent heat-source results obtained with exact data ($* \ * \ *$) and when using zero-order Tikhonov regularization for $p = 1$ ($\cdots$), $p = 3$ (– – –), and $p = 5$ (–·· – ··–) percent noise, in comparison with the exact solution (———) for Example 1.

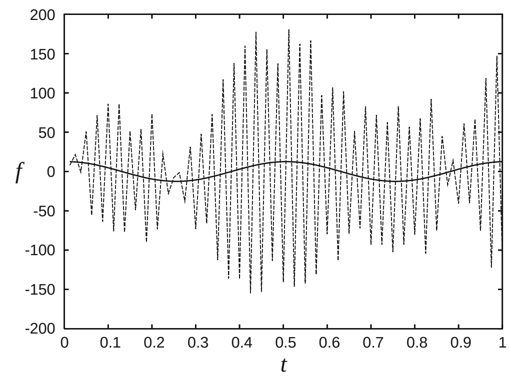

Figure 2. The numerical time-dependent heat source results obtained with noisy data with $p = 3$ and no regularization (– – –), in comparison with the exact solution (———) for Example 1.

4.1. Example for problem I

With the input data

$$u(0,t) = h_0(t) = 2t + \sin(4\pi t), \quad u(1,t) = h_1(t) = 1 + 2t + \sin(4\pi t), \tag{75}$$

$$u(x,0) = u_0(x) = x^2, \quad u(0.5,t) = \chi(t) = 0.25 + 2t + \sin(4\pi t), \tag{76}$$

the inverse problem (1)–(4) has the unique solution given by

$$u(x,t) = x^2 + 2t + \sin(4\pi t), \tag{77}$$

$$f(t) = 4\pi \cos(4\pi t). \tag{78}$$

Figure 1 shows the numerical results obtained for estimating the timewise heat source (78) when employing the Tikhonov zero-order regularization and when both exact and noisy data was used. It can be seen from Figure 1 that the numerical solution obtained in the ideal case, when no noise is contained in the input data (75)–(76), is graphically almost indistinguishable from the analytical solution (78). In fact, the regularization parameter in this case was chosen to be $\lambda = 0$, which is equivalent to saying that no regularization is needed in this case and the numerical solution can be obtained using a standard Gaussian elimination technique.

Next, the input data (75)–(76) was perturbed by $p \in \{1, 3, 5\}$ percent random Gaussian additive noise. Figure 2 shows the results obtained with $p = 3$ percent noise included in the data and no regularization. It can be observed in this case that the numerical solution is highly oscillatory and contains errors which are more than about one order of magnitude larger than the analytical solution. In order to overcome this instability we use next the Tikhonov regularization.

From the numerical results shown in Figure 1 it can be seen that the numerical solution obtained using the zero-order Tikhonov regularization technique is converging to the exact solution (78), as the amount of noise p decreases. Also, it can be seen from Figure 1 that there are some inaccuracies in these numerical solutions. The inaccuracies are clearly visible

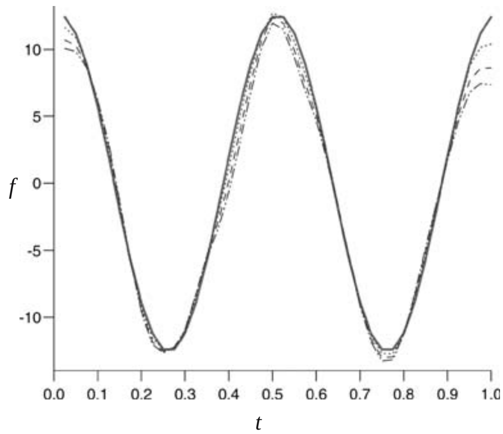

Figure 3. The numerical time-dependent heat-source results obtained when using first-order Tikhonov regularization for $p=1$ $(\cdots)$, $p=3(-\ -\ -)$, and $p=5(-\cdot\cdot - \cdot\cdot-)$ percent noise, in comparison with the exact solution (———) for Example 1.

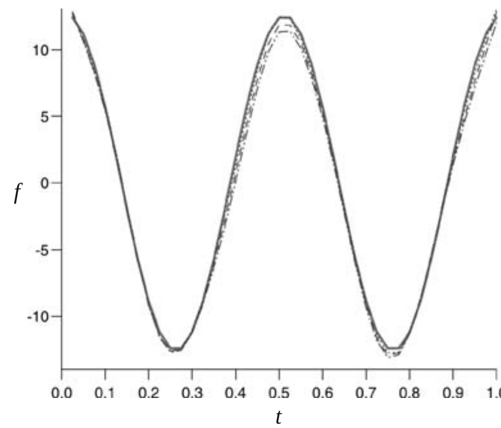

Figure 4. The numerical time-dependent heat-source results obtained when using second order Tikhonov regularization for $p=1(\cdots)$, $p=3(-\ -\ -)$, and $p=5(-\cdot\cdot - \cdot\cdot-)$ percent noise, in comparison with the exact solution (———) for Example 1.

Table 1. The values of the regularization parameters λ used for Example 1.

Tikhonov order	$p=1\%$	$p=3\%$	$p=5\%$
Zero	0·00013	0·0004	0·00065
First	0·00050	0·0016	0·0025
Second	0·0015	0·0079	0·0125

at the endpoints of the time interval, where the convergence of the numerical solution is particularly slow.

Figure 3 shows the numerical results obtained using the same sets of noisy data as previously, but when first-order Tikhonov regularization was employed. The results are now clearly improved in accuracy when compared with the zero-order results of Figure 1, although some inaccuracies can still be seen towards the solution endpoints.

Finally, the most accurate results for this example were obtained when using second-order Tikhonov regularization and they are presented in Figure 4. The same amounts of random Gaussian noise as in the previous two cases were added to the input data (75)–(76), *i.e.*, $p\in\{1,3,5\}$ percent. It can be seen from Figure 4 that the numerical solutions are all stable and very close to the analytical solution and that, as the amount of noise p decreases, they are converging to the exact solution (78).

The values of the regularization parameter λ were chosen according to the discrepancy principle and they are shown in Table 1.

4.2. Example for problem II

With the input data

$$u(0,t)=h_0(t)=0, \quad u(1,t)=h_1(t)=0, \tag{79}$$

$$u(x,0)=u_0(x)=\sin(\pi x), \quad u(x,1)=\Psi(x)=(2-e^{-\pi^2})\sin(\pi x) \tag{80}$$

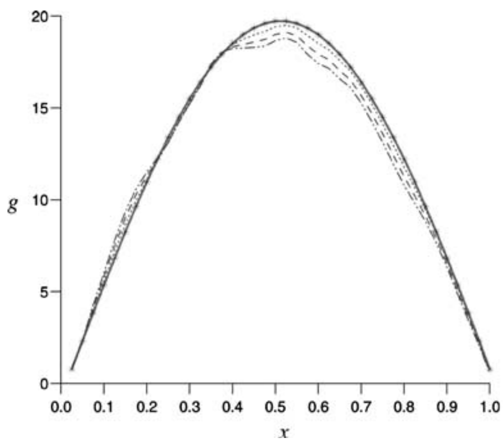

Figure 5. The numerical space-dependent heat source results obtained with exact data (∗ ∗ ∗) and when using zero-order Tikhonov regularization for $p = 1(\cdots)$, $p = 3(-\ -\ -)$, and $p = 5$ (—·· — ··—) percent noise, in comparison with the exact solution (———) for Example 2.

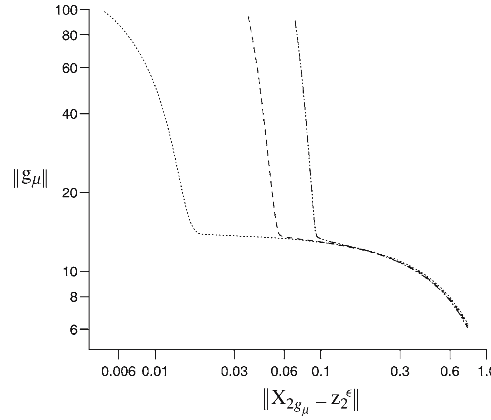

Figure 6. The L-curves obtained for $p = 1(\cdots)$, $p = 3(-\ -\ -)$, and $p = 5$(—·· — ··—) percent noise, for Example 2.

the inverse problem (2), (3), (8) and (9) has the unique solution given by

$$u(x, t) = (2 - e^{-\pi^2 t}) \sin(\pi x), \tag{81}$$

$$g(x) = 2\pi^2 \sin(\pi x). \tag{82}$$

Figure 5 shows the numerical results obtained for estimating the spacewise heat source (82) when exact data was used, *i.e.*, no noise was included in the input data (79)–(80), and the Tikhonov zero-order regularization technique was employed. It can be seen from Figure 5 that the numerical solution obtained in this ideal case is graphically indistinguishable from the analytical solution (82). Next, the input data (79)–(80) was perturbed by $p \in \{1, 3, 5\}$ percent random Gaussian additive noise and the numerically obtained results are also shown in Figure 5. The regularization parameters μ were chosen according to the L-curve criterion and their values were $\mu = 8 \times 10^{-5}$ for $p = 1\%$, $\mu = 1{\cdot}5 \times 10^{-4}$ for $p = 3\%$ and $\mu = 2 \times 10^{-4}$ for $p = 5\%$ and these values were in good agreement with those suggested by the discrepancy principle. The L-curves obtained for Example 2 are shown in Figure 6.

From Figure 5 it can be seen that all the numerical solutions are stable and that, as the amount of noise p decreases, the numerical solution converges to the exact solution (82). We wish to report that we have employed not only the zero-order regularization, but also the first- and second-order Tikhonov regularizations. However, the numerical results obtained with these higher-order regularizations showed a similar degree of accuracy as the zero-order regularization and thus they are not explicitly presented here. It should be observed that, as also mentioned in Section 3, these higher-order regularizations, as opposed to zero-order one, are imposing a smoothness condition on either the first- or second-order derivative instead of the solution itself. However, these smoother, higher-order solutions are not necessarily more accurate than the zero-order solution.

5. Conclusions

In this paper a boundary-element method combined with a regularization technique has been developed for obtaining stable timewise or spacewise dependent heat sources, from over-specified conditions which ensure unique solvability for the inverse and ill-posed heat-source problems. Numerical results were presented for two inverse heat-source problems which had the input data perturbed by increasing amounts of random noise. Various orders of Tikhonov regularization have been employed and discussed and the choice of the regularization parameter was based on the discrepancy and L-curve principles. The obtained results show that the numerical solutions are stable and converge to the exact solution as the amount of noise added to the input data decreases.

Although we have only considered Dirichlet boundary conditions, there are also solvability theorems for the problem of finding a source for the parabolic heat equation with general boundary conditions. A similar approach can be used for the determination of a single variable source for the heat-conduction equation in another inverse formulation, with an integral overdetermination condition specifying the energy variation of the heat-conducting system; see [4].

The study performed in this paper can be extended to higher dimensional parabolic partial differential equations of order n with constant coefficients $(a_i)_{i=\overline{1,n}}$, b and $(k_{ij})_{i,j=\overline{1,n}}$ positive definite, of the form

$$\sum_{i,j=1}^{n} k_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + bu = \frac{\partial u}{\partial t} + Q(\mathbf{x}, t) \tag{83}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$, $t \in (0, T)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain and the unknown heat source $Q(\mathbf{x}, t)$ is independent of the space or time variable, [7].

Future work will be concerned with the numerical determination of a source $Q(x,t) = f(t) + g(x)$ or $Q(x,t) = f(t)g(x)$ which is both space- and time-dependent, but additive or separable, from measurements of the temperature at a given single instant or at two instants and two interior locations. Other possible future work may also concern the numerical determination using the dual reciprocity BEM of a source of the type $Q(x,t) = f(t)u$, [16], $Q(x,t) = f(t)u + q(t)\frac{\partial u}{\partial x}$, [17], or $Q = F(u)$, [18].

Acknowledgement

The authors would like to thank T. Johansson for his useful comments on this work.

Appendix A. Auxiliary results

In this appendix we first give the notations from [10, pp. 60–74], for the spaces involved in Theorems 1–4. Let us introduce the following notation:

$$\langle u\rangle_{\Omega}^{\alpha} = \frac{u(x) - u(y)}{|x-y|^{\alpha}}, \tag{A1}$$

for a function u depending on $x \in \mathbb{R}^n$. Here, $0 < \alpha < 1$ and $x, y \in \overline{\Omega} \subseteq \mathbb{R}$. Moreover, we define

$$\langle u\rangle_{x,Q_T}^{\alpha} = \frac{u(x,t_0) - u(y,t_0)}{|x-y|^{\alpha}}, \tag{A2}$$

$$\langle u\rangle_{t,Q_T}^{\alpha} = \frac{u(x,t_1) - u(x,t_0)}{|t_1-t_0|^{\alpha}}, \tag{A3}$$

for a function depending on $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Here, $Q_T = \Omega \times (0, T), 0 < \alpha < 1$, and $(x, t_0), (x, t_1), (y, t_0) \in \overline{Q}_T$. If Ω is sufficiently smooth, the Holder space $H^l(\overline{\Omega})$ can be described as consisting of functions $u(x)$ with continuous derivatives of order less or equal with l, up to $\overline{\Omega}$, with the finite norm

$$\|u\|_{H^l(\overline{\Omega})} = \sum_{|\alpha|=[l]} \langle \partial_x^\alpha u \rangle_{\overline{\Omega}}^{(l-[l])} + \sum_{|\alpha| \le l} \max_{x \in \overline{\Omega}} |\partial_x^\alpha u|. \tag{A4}$$

The space $H^{l,l/2}(\overline{Q}_T)$ consists of functions $u(x,t)$ with continuous derivatives on $\overline{Q}_T$ of order less than l, and with the finite norm

$$\|u\|_{H^{l,l/2}(\overline{Q}_T)} = \sum_{|\overline{\alpha}|=[l]} \langle \partial_t^{\alpha_0} \partial_x^\alpha u \rangle_{x,\overline{Q}_T}^{(l-[l])} + \sum_{0<l-|\overline{\alpha}|<2} \langle \partial_t^{\alpha_0} \partial_x^\alpha u \rangle_{t,\overline{Q}_T}^{(l-|\overline{\alpha}|)/2} + \sum_{|\overline{\alpha}| \le l} \max_{(x,t)\in\overline{Q}_T} |\partial_t^{\alpha_0} \partial_x^\alpha u| \tag{A5}$$

Here $\overline{\alpha} = (\alpha, \alpha_0)$ and $|\overline{\alpha}| = 2\alpha_0 + \alpha$.

We give now the integrals of G and $\partial G/\partial n$ which arise from the discretization of the boundary-integral equation (22) using the BEM:

$$A_{\xi j}(x,t) = \int_{t_{j-1}}^{t_j} G(x,t,\xi,\tau)\mathrm{d}\tau = \begin{cases} 0, & t \le t_{j-1} \\ \sqrt{(t-t_{j-1})/\pi}, & t_{j-1} < t \le t_j,\ r=0 \\ r(\exp(-z^2)/z - \sqrt{\pi}\,\mathrm{erfc}(z))/(2\sqrt{\pi}), & t_{j-1} < t \le t_j,\ r \ne 0 \\ r\{\exp(-z^2)/z - \exp(-z_1^2)/z_1 & \\ \quad + \sqrt{\pi}(\mathrm{erf}(z) - \mathrm{erf}(z_1))\}/(2\sqrt{\pi}), & t > t_j\ , \end{cases} \tag{A6}$$

$$B_{\xi j}(x,t) = \int_{t_{j-1}}^{t_j} \frac{\partial G}{\partial n(\xi)}(x,t,\xi,\tau)\mathrm{d}\tau = \begin{cases} 0, & t \le t_{j-1} \\ 0, & t_{j-1} < t \le t_j,\ r=0 \\ -\mathrm{erfc}(z)/2, & t_{j-1} < t \le t_j,\ r \ne 0 \\ (\mathrm{erf}(z) - \mathrm{erf}(z_1))/2, & t > t_j \end{cases} \tag{A7}$$

with $\xi \in \{0, l\}, r = |x - \xi\,|, z = r[(t-t_{j-1})]^{-1/2}/2, z_1 = r[(t-t_j)]^{-1/2}/2,$

$$C_k(x,t) = \int_{x_{k-1}}^{x_k} G(x,t,y,0)\mathrm{d}y = \frac{1}{2}\left[\mathrm{erf}\left(\frac{x-x_{k-1}}{2\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x-x_k}{2\sqrt{t}}\right)\right], \tag{A8}$$

$$D_j^I(x,t) = \int_{t_{j-1}}^{t_j}\int_0^l G(x,t,y,\tau)\mathrm{d}y\mathrm{d}\tau = \begin{cases} 0, & t \le t_{j-1} \\ -\frac{2x^2-2x+1}{4} - J_1(x,t,t_{j-1}), & t_{j-1} < t < t_j\ , \\ J_1(x,t,t_j) - J_1(x,t,t_{j-1}), & t_j \le t \end{cases} \tag{A9}$$

$$\begin{aligned} J_1(x,t,t_0) = &-\frac{r}{2}\mathrm{erf}(z) - \frac{1}{2\sqrt{\pi}}\frac{x\sqrt{r}}{\exp(z^2)} - \frac{x^2}{4}\mathrm{erf}(z) \\ &+\frac{r}{2}\mathrm{erf}(z_1) + \frac{1}{\sqrt{\pi}}\frac{(x-l)\sqrt{r}}{2\exp(z^2)} + \frac{(x-l)^2}{4}\mathrm{erf}(z_1) \end{aligned} \tag{A10}$$

with $r = t - t_0, z = \frac{x}{2\sqrt{t-t_0}}$ and $z_1 = \frac{x-l}{2\sqrt{t-t_0}}$,

$$D_k^{II}(x,t) = \int_{x_{k-1}}^{x_k}\int_0^t G(x,t,y,\tau)\mathrm{d}\tau\mathrm{d}y = J_2(x,t,x_k) - J_2(x,t,x_{k-1}), \tag{A11}$$

$$J_2(x,t,x_0) = -t(1+z^2)\mathrm{erf}(z) + \frac{2}{\sqrt{\pi}}tz + \frac{1}{4}\begin{cases} x_0^2 - 2xx_0, & x_0 \le x \\ 2xx_0 - 2x^2 - x_0^2, & x < x_0 \end{cases} \tag{A12}$$

with $z = \frac{x-x_0}{2\sqrt{t}}$.

In the above expressions, $x \in [0, 1]$, $t \in (0, 1]$, and erf and erfc are the error functions defined as

$$\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} \mathrm{e}^{-\sigma^2} \mathrm{d}\sigma, \quad \mathrm{erfc}(\xi) = 1 - \mathrm{erf}(\xi). \tag{A13}$$

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